

# The ILW hierarchy

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## Abstract

In this paper, we present a new hierarchy which includes the intermediate long wave (ILW) equation at the lowest order. This hierarchy is thought to be a novel reduction of the 1st modified KP type hierarchy. The framework of our investigation is Sato theory.

## 1 Introduction

It is well known that Sato theory was established by M.Sato around 1980 to give a unified viewpoint for integrable soliton equations [1]. A lot of important studies have been based on this magnificent theory since then. In this paper, we focus on one of the basic ideas of the theory, summarized as follows [2].

“Start from an ordinary differential equation and suppose that the solutions satisfy certain dispersion relations. Then, as conditions the coefficients must satisfy, we obtain a set of nonlinear partial differential equations. If we assume a particular set of linear dispersion relations, we obtain the KP hierarchy as the corresponding PDEs.”

It is known that most nonlinear partial differential equations which have  $N$ -soliton solutions correspond to certain equations of KP hierarchy. But, in the case of the intermediate long wave (ILW) equation, although its  $N$ -soliton solutions and an inverse scattering transform were well-known [3,4,5], the correspondence to the KP hierarchy has remained unclear.

The ILW equation was proposed by Joseph [6] and Kubota et al. [7] to describe long internal gravity waves in a stratified fluid with finite depth. It is written in the form

$$u_t + \frac{1}{\delta} u_x + 2uu_x + T(u_{xx}) = 0, \quad (1)$$

where  $T(\cdot)$  is the singular integral operator given by

$$T(u) = P \int_{-\infty}^{\infty} \frac{1}{2\delta} \cot \left[ \frac{\pi}{2\delta} (\xi - x) \right] u(\xi) d\xi \quad (2)$$

( $P$  represents the principal value of the integral). Depending on the parameter  $\delta$  (which controls the depth of the internal wave layer) (1) reduces to the Korteweg-de Vries (KdV) equation as  $\delta \rightarrow 0$  or to the Benjamin-Ono (BO) equation as  $\delta \rightarrow \infty$ .

In this paper, we present a new hierarchy which includes the ILW equation at the lowest order. This hierarchy is thought to be a novel reduction of the 1st modified KP type hierarchy. In section 2, we propose a set of differential-difference dispersion relations, and introduce a corresponding 1st modified KP type hierarchy. In section 3, this 2 + 1dimensional hierarchy will be reduced to a 1 + 1dimensional hierarchy which contains the ILW equation at its lowest order. In section 4, we discuss more general dispersion relations of differential-difference type.

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## 2 The dispersion relations and their corresponding hierarchy

Let us introduce a pseudo-differential operator,

$$W = 1 - \frac{U}{2}\partial^{-1} + w_2\partial^{-2} + w_3\partial^{-3} + w_4\partial^{-4} + \cdots + w_m\partial^{-m}, \quad (3)$$

where  $U$  and  $w_j$  ( $j = 1, 2, \dots$ ) are functions of continuous variables  $t = (t_1, t_2, \dots)$  and a discrete variable  $z$ . We sometimes use  $x$  instead of  $t_1$  to follow the convention.  $\partial^{-n}$  denotes

$$\partial^{-n} = \left( \frac{d}{dt_1} \right)^{-n}. \quad (4)$$

If we employ the Leibniz rule,

$$\partial^n f(t_1) = \sum_{r=0}^{\infty} \frac{n(n-1)\cdots(n-r+1)}{r!} \left( \frac{\partial^r f}{\partial t_1^r} \right) \partial^{n-r}, \quad (5)$$

then  $\partial^n$  can be a well-defined operator even for negative integer  $n$ . Though the theory is developed for the case of  $m \rightarrow \infty$  in general, we, in this paper, confine ourselves to (3) for simplicity [2]. It is remarked that the essence of the general theory is still kept in this simplification.

Let us consider the ordinary differential equation,

$$W\partial^m f(t, z) = (\partial^m - \frac{U}{2}\partial^{m-1} + w_2\partial^{m-2} + \cdots + w_m)\partial^m f(t, z) = 0 \quad (6)$$

which has  $m$  linearly independent solutions,  $f^{(1)}(t, z), f^{(2)}(t, z), \dots, f^{(m)}(t, z)$ . We assume here that  $f^{(1)}(t, z), f^{(2)}(t, z), \dots, f^{(m)}(t, z)$  satisfy the following dispersion relations,

$$\begin{cases} i^{1-n}\partial_{t_n} f^{(j)} = \partial^n f^{(j)} \\ \Delta_z f^{(j)} = \partial f^{(j)} \end{cases} \quad (j = 1, 2, \dots, m \text{ and } n = 1, 2, \dots), \quad (7)$$

where  $\Delta_z$  denotes a difference operator,

$$\begin{aligned} \Delta_z g(z) &= \frac{e^{2i\delta\partial_z} - 1}{2i\delta} g(z) \\ &= \frac{g(z + 2i\delta) - g(z)}{2i\delta} \end{aligned} \quad (8)$$

( $\delta$  is an arbitrary constant).

It should be remarked that  $U$  and  $w_j$  are expressible by means of a  $\tau$ -function, which is the Wronskian of  $f^{(1)}(t, z), f^{(2)}(t, z), \dots, f^{(m)}(t, z)$ , i.e.

$$U = 2\frac{\tau_x}{\tau}, \quad w_2 = \frac{1}{2} \left( \frac{\tau_{t_2}}{\tau} - \frac{\tau_{xx}}{\tau} \right), \dots \quad (9)$$

Let  $B_n$  ( $n = 1, 2, \dots$ ) and  $C$  be pseudo-differential operators,

$$B_n = (W\partial^n W^{-1})_+, \quad (10)$$

$$C = (\bar{W}\partial W^{-1})_+, \quad (11)$$

where  $\bar{W}(z) = W(z + 2i\delta)$  and  $(A)_+$  denotes the differential part of the pseudo differential operator  $A$ . We use  $\bar{\cdot}$  to denote a shift operator:  $z \rightarrow z + 2i\delta$ . Then we introduce time evolution equations for  $W$  by

$$\begin{cases} i^{1-n}\frac{\partial W}{\partial t_n} = B_n W - W\partial^n \\ \frac{\bar{W} - W}{2i\delta} = CW - \bar{W}\partial \end{cases} \quad (n = 1, 2, \dots), \quad (12)$$

which are Sato type equations. From (12) we get an infinite system of the Zakharov-Shabat type equations

$$\begin{cases} i^{1-l}(C)_{t_l} - \frac{\bar{B}_l - B_l}{2i\delta} + CB_l - \bar{B}_l C = 0 \\ i^{1-l}(B_k)_{t_l} - i^{1-k}(B_l)_{t_k} + [B_k, B_l] = 0 \end{cases} \quad (13)$$

$(k, l = 1, 2, \dots).$

Furthermore from (13) we can deduce a system of partial differential-difference equations for  $U$ . The first few are explicitly given by

$$i(\bar{U} - U)_{t_2} + \frac{i}{\delta}(\bar{U} - U)_x + (\bar{U} - U)(\bar{U} - U)_x + (\bar{U} + U)_{xx} = 0, \quad (14)$$

$$\begin{aligned} 2(\bar{U} - U)_{t_3} - \frac{3i}{2}(\bar{U} - U)_{xt_2} + \frac{1}{2}(\bar{U} - U)_{xxx} \\ + \frac{3i}{2\delta}(\bar{U} + U)_{xx} + \frac{6i}{2\delta}(\bar{U} - U)(\bar{U} - U)_x - \frac{3}{2\delta^2}(\bar{U} - U)_x \\ + \frac{3}{2}(\bar{U} + U)_{xx}(\bar{U} - U) + \frac{3}{2}(\bar{U} + U)_x(\bar{U} - U)_x + \frac{3}{2}(\bar{U} - U)^2(\bar{U} - U)_x = 0, \end{aligned} \quad (15)$$

$$\begin{aligned} -i(\bar{U} - U)_{t_4} - \frac{1}{2}(\bar{U} + U)_{t_2 t_2} + \frac{i}{2}(\bar{U} - U)_{xxt_2} + (\bar{U} + U)_{xxxx} \\ + 3(\bar{U} - U)_x(\bar{U} - U)_{xx} - \frac{i}{2}(\bar{U} + U)_{t_2}(\bar{U} - U)_x - i(\bar{U} - U)_{t_2}(\bar{U} + U)_x \\ - \frac{i}{2}(\bar{U} + U)_{xt_2}(\bar{U} - U) + \frac{i}{2\delta^2}(\bar{U} - U)_{t_2} + \frac{1}{2\delta}(\bar{U} + U)_{xt_2} + \frac{i}{\delta}(\bar{U} - U)_{xxx} \\ + \frac{1}{\delta}(\bar{U} - U)(\bar{U} - U)_{t_2} + (\bar{U} - U)(\bar{U} - U)_{xxx} - \frac{i}{2}(\bar{U} - U)^2(\bar{U} - U)_{t_2} = 0. \end{aligned} \quad (16)$$

Substituting (9) into (14)-(16), we also have the equations for  $\tau$  by

$$\left(iD_{t_2} + \frac{i}{\delta}D_x + D_x^2\right)\bar{\tau} \cdot \tau = 0, \quad (17)$$

$$\left(4D_{t_3} - 3iD_x D_{t_2} + D_x^3 + \frac{3i}{\delta}D_x^2 - \frac{3}{\delta^2}D_x\right)\bar{\tau} \cdot \tau = 0, \quad (18)$$

$$\left(-2iD_{t_4} - D_{t_2}^2 - iD_{t_2}D_x^2 + \frac{1}{\delta}D_{t_2}D_x + \frac{i}{\delta^2}D_{t_2}\right)\bar{\tau} \cdot \tau = 0 \quad (19)$$

( $D$  denotes Hirota's differential operator). It should be remarked that (17)-(19) are essentially the same as the first few of the 1st modified KP hierarchy [8],

$$(D_{t_1}^2 + D_{t_2})\tau_n \cdot \tau_{n+1} = 0, \quad (20)$$

$$(D_{t_1}^3 - 4D_{t_3} - 3D_{t_1}D_{t_2})\tau_n \cdot \tau_{n+1} = 0, \quad (21)$$

$$(-D_{t_1}^2D_{t_2} + D_{t_2}^2 + 2D_{t_4})\tau_n \cdot \tau_{n+1} = 0. \quad (22)$$

### 3 Solutions and special reductions

$N$ -soliton solutions for (17)-(19) will be written in the form

$$\begin{aligned} \tau = & \left| \begin{array}{ccc} 1 + c_1 e^{\eta(t, p_1) - \eta(t, q_1)} & \cdots & 1 + c_N e^{\eta(t, p_N) + \eta(t, q_N)} \\ l(q_1) + l(p_1)c_1 e^{\eta(t, p_1) - \eta(t, q_1)} & \cdots & l(q_N) + l(p_N)c_N e^{\eta(t, p_N) - \eta(t, q_N)} \\ \vdots & \cdots & \vdots \\ l(q_1)^{N-1} + l(p_1)^{N-1}c_1 e^{\eta(t, p_1) - \eta(t, q_1)} & \cdots & l(q_N)^{N-1} + l(p_N)^{N-1}c_N e^{\eta(t, p_N) - \eta(t, q_N)} \end{array} \right| \\ & / \prod_{j' > j} (l(q_{j'}) - l(q_j)) \end{aligned}$$

$$= \sum_{J \subset I} \left( \prod_{j \in J} c_j \right) \left( \prod_{\substack{j, j' \in J \\ j < j'}} a_{jj'} \right) \exp \left( \sum_{j \in J} \eta(t, p_j) - \eta(t, q_j) \right), \quad (23)$$

where the summation is taken over all subsets  $J$  of  $I = \{1, 2, \dots, m\}$ .  $\eta(t, p)$ ,  $l(p)$  and  $a_{jj'}$  are defined by

$$\eta(t, p) = pz + \sum_{n=1}^{\infty} i^{n-1} l(p)^n t_n, \quad (24)$$

$$l(p) = \frac{e^{2i\delta p} - 1}{2i\delta}, \quad (25)$$

$$a_{jj'} = \frac{l(p_j) - l(p_{j'})}{l(p_j) - l(q_{j'})} \cdot \frac{l(q_j) - l(q_{j'})}{l(q_j) - l(p_{j'})}, \quad (26)$$

and the  $c_j$  are constants. Notably, the bilinear identity to  $\tau$  is given as follows.

**LEMMA 3.1** *For arbitrary  $t = (t_1, t_2, \dots)$ ,  $t' = (t'_1, t'_2, \dots)$  and  $z$ ,  $\tau$  satisfies*

$$\oint e^{-i\xi(k, t-t')} (1 + 2\delta k) \tau(z + 2i\delta, t - i\epsilon(k^{-1})) \tau(z, t + i\epsilon(k^{-1})) \frac{dk}{2\pi i} = 0, \quad (27)$$

where  $\xi(k, t) = \sum_{n=0}^{\infty} t_n k^n$ ,  $\epsilon(k^{-1}) = (\frac{1}{k}, \frac{1}{2k^2}, \dots, \frac{1}{nk^n}, \dots)$ . The curve is taken around  $\infty$  and excludes the singular points  $il(p)$ ,  $il(q)$ .

*Proof.* Substitute (23) into (27) and we see that  $\text{Res}(il(p_n)) + \text{Res}(il(q_n)) = 0$  for  $\forall n$ .  $\square$

We impose the condition

$$p_j - q_j = l(p_j) - l(q_j) = k_j \quad (28)$$

on (23). Then, it is reduced to

$$\tau = \sum_{J \subset I} \left( \prod_{j \in J} c_j \right) \left( \prod_{\substack{j, j' \in J \\ j < j'}} a_{jj'} \right) \exp \left( \sum_{j \in J} \left( k_j z + \sum_{n=1}^{\infty} \mu_n(k_j) t_n \right) \right) \quad (29)$$

for appropriately-defined functions  $\mu_n(k)$ . It should be noticed that (29) is the same as the soliton solution of the ILW equation [9]. We here present the 1-soliton solution as an example.

$$\begin{aligned} \tau = 1 + c \exp & \left[ kz + kt_1 + \left( k^2 \cot k\delta - \frac{k}{\delta} \right) t_2 \right. \\ & + \frac{1}{4} \left( k^3 - 3k^3 \cot^2 k\delta + \frac{6}{\delta} k^2 \cot k\delta - \frac{3}{\delta^2} k \right) t_3 \\ & \left. + \frac{1}{2} \left( -k^4 \cot k\delta + k^4 \cot^3 k\delta - \frac{3}{\delta} k^3 \cot^2 k\delta + \frac{1}{\delta} k^3 + \frac{3}{\delta^2} k^2 \cot k\delta - \frac{1}{\delta^3} k \right) t_4 + \dots \right] \end{aligned} \quad (30)$$

We can regard  $z$  and  $t_1$  as the same variable under this reduction, because the coefficient of  $z$  is equal to that of  $t_1$  in the exponentiated part of (29). It also should be noticed that

**LEMMA 3.2** *If  $k_j$ 's are real,  $\mu_n(k_j)$  and  $a_{jj'}$  are also real ( $j, n = 1, 2, \dots$ ).*

*Proof.*

From (25) and (28), we get

$$l(p_j) + l(q_j) = -ik_j \cot k_j \delta + \frac{i}{\delta}. \quad (31)$$

Because  $l(p_j) + l(q_j)$  is purely imaginary and  $l(p_j) - l(q_j) (= k_j)$  is real, there exist real  $r, \theta$  by which

$$l(p_j) = re^{i\theta}, \quad l(q_j) = re^{i(\pi-\theta)}. \quad (32)$$

By the definition of  $\mu_n(k_j)$ , we have

$$\begin{aligned}
\mu_n(k_j) &= i^{n-1} (l(p_j)^n - l(q_j)^n) \\
&= i^{n-1} \left( r e^{in\theta} - r e^{in(\pi-\theta)} \right) \\
&= \begin{cases} 2i^{n-1} r^n \cos(n\theta) & \text{for } n \text{ odd} \\ 2i^n r^n \sin(n\theta) & \text{for } n \text{ even.} \end{cases}
\end{aligned} \tag{33}$$

Hence  $\mu_n(k_j)$  is real. We also deduce

$$\begin{aligned}
a_{jj'} &= \frac{l(p_j) - l(p_{j'})}{l(p_j) - l(q_{j'})} \cdot \frac{l(q_j) - l(q_{j'})}{l(q_j) - l(p_{j'})} \\
&= \frac{(r e^{i\theta} - r' e^{i\theta'})(r e^{-i\theta} - r' e^{-i\theta'})}{(r e^{i\theta} - r' e^{i\theta'})(r e^{i\theta} - r' e^{i\theta'})} \\
&= \frac{r^2 + r'^2 - r r' \cos(\theta - \theta')}{r^2 + r'^2 - r r' \cos(\theta + \theta')},
\end{aligned} \tag{34}$$

which gives that  $a_{jj'}$  is real.  $\square$

By this reduction, the  $x$ -shifts can take the place of the  $z$ -shifts and (14)-(16) are rewritten into the equations for  $U^\pm(x) := U(x \mp i\delta)$ , i.e.

$$i(U^- - U^+)_{t_2} + \frac{i}{\delta}(U^- - U^+)_x + (U^- - U^+)(U^- - U^+)_x + (U^- + U^+)_{xx} = 0, \tag{35}$$

$$\begin{aligned}
2(U^- - U^+)_{t_3} - \frac{3i}{2}(U^- - U^+)_{xt_2} + \frac{1}{2}(U^- - U^+)_{xxx} + \frac{3i}{2\delta}(U^- + U^+)_{xx} \\
+ \frac{3i}{\delta}(U^- - U^+)(U^- - U^+)_x - \frac{3}{2\delta^2}(U^- - U^+)_x + \frac{3}{2}(U^- + U^+)_{xx}(U^- - U^+) \\
+ \frac{3}{2}(U^- + U^+)_x(U^- - U^+)_x + \frac{3}{2}(U^- - U^+)^2(U^- - U^+)_x = 0,
\end{aligned} \tag{36}$$

$$\begin{aligned}
-i(U^- - U^+)_{t_4} - \frac{1}{2}(U^- + U^+)_{t_2 t_2} + \frac{i}{2}(U^- - U^+)_{xxt_2} + (U^- + U^+)_{xxxx} \\
+ 3(U^- - U^+)_x(U^- - U^+)_{xx} - \frac{i}{2}(U^- + U^+)_{t_2}(U^- - U^+)_x - i(U^- - U^+)_{t_2}(U^- + U^+)_x \\
- \frac{i}{2}(U^- + U^+)_{xt_2}(U^- - U^+) + \frac{i}{2\delta^2}(U^- - U^+)_{t_2} + \frac{1}{2\delta}(U^- + U^+)_{xt_2} + \frac{i}{\delta}(U^- - U^+)_{xxx} \\
+ \frac{1}{\delta}(U^- - U^+)(U^- - U^+)_{t_2} + (U^- - U^+)(U^- - U^+)_{xxx} - \frac{i}{2}(U^- - U^+)^2(U^- - U^+)_{t_2} = 0.
\end{aligned} \tag{37}$$

If we consider lemma 3.2 and suppose that  $U(x)$  is analytic in the horizontal strip between  $\text{Im } x = -i\delta$  and  $\text{Im } x = i\delta$ , we can introduce a dependent variable  $u$  which satisfies [9]

$$u = \frac{i}{2}(U^- - U^+), \tag{38}$$

$$T(u) = \frac{1}{2}(U^- + U^+). \tag{39}$$

Substituting this  $u$  into (35)-(37), we obtain

$$u_{t_2} + \frac{1}{\delta}u_x + 2uu_x + T(u_{xx}) = 0, \tag{40}$$

$$\begin{aligned}
-4u_{t_3} - 3T(u_{xt_2}) - u_{xxx} - 6uT(u_{xx}) - 6u_xT(u_x) + 12u^2u_x \\
- \frac{12}{\delta}uu_x + \frac{3}{\delta}T(u_{xx}) + \frac{3}{\delta^2}u_x = 0,
\end{aligned} \tag{41}$$

$$\begin{aligned}
-2u_{t_4} - T(u_{t_2 t_2}) + u_{xxt_2} - 4u^2u_{t_2} - 2u_xT(u_{t_2}) - 4u_{t_2}T(u_x) \\
- 2uT(u_{xt_2}) + 2T(u_{xxx}) - 12u_xu_{xx} - 4uu_{xxx} \\
+ \frac{1}{\delta}T(u_{xt_2}) + \frac{2}{\delta}u_{xxx} - \frac{4}{\delta}uu_{t_2} + \frac{1}{\delta^2}u_{t_2} = 0.
\end{aligned} \tag{42}$$

Because the lowest order is nothing but the intermediate long wave equation, this hierarchy should be called the ILW hierarchy.

## 4 More general dispersion relations

Other possible differential-difference dispersion relations than (7) can be

$$\begin{cases} i^{1-n} \partial_{t_n} f^{(j)} = \partial^n f^{(j)} \\ i^{s-1} \Delta_z f^{(j)} = \partial^s f^{(j)}, \end{cases} \quad (43)$$

where  $s$  is some fixed positive integer,  $j = 1, 2, \dots, m$  and  $n = 1, 2, \dots$ . Notably, (43) corresponds to (7) when  $s = 1$ .

For each  $s$ , we can deduce the corresponding hierarchy in the same way as in the preceding sections. Hence we just present the bilinear identity which generates the hierarchy for  $\tau$  [8],

$$\oint e^{-i\xi(k, t-t')} (1 - 2\delta(-k)^s) \tau(z + 2i\delta, t - i\epsilon(k^{-1})) \tau(z, t + i\epsilon(k^{-1})) \frac{dk}{2\pi i} = 0. \quad (44)$$

The  $N$ -soliton solution to the hierarchy is written in the form

$$\begin{aligned} \tau &= \left| \begin{array}{ccc} 1 + c_1 e^{H(t, p_1) - H(t, q_1)} & \dots & 1 + c_N e^{H(t, p_N) + H(t, q_N)} \\ L(q_1) + L(p_1) c_1 e^{H(t, p_1) - H(t, q_1)} & \dots & L(q_N) + L(p_N) c_N e^{H(t, p_N) - H(t, q_N)} \\ \vdots & \dots & \vdots \\ L(q_1)^{N-1} + L(p_1)^{N-1} c_1 e^{H(t, p_1) - H(t, q_1)} & \dots & L(q_N)^{N-1} + L(p_N)^{N-1} c_N e^{H(t, p_N) - H(t, q_N)} \end{array} \right| \\ &/ \prod_{j' > j} (L(q_{j'}) - L(q_j)) \\ &= \sum_{J \subset I} \left( \prod_{j \in J} C_j \right) \left( \prod_{\substack{j, j' \in J \\ j < j'}} A_{jj'} \right) \exp \left( \sum_{j \in J} H(t, p_j) - H(t, q_j) \right), \end{aligned} \quad (45)$$

where the summation is taken over all subsets  $J$  of  $I = \{1, 2, \dots, m\}$ .  $H(t, p), L(p)$  and  $A_{jj'}$  are defined by

$$H(t, p) = pz + \sum_{n=1}^{\infty} i^{n-1} L(p)^n t_n, \quad (46)$$

$$i^{1-s} L(p)^s = \frac{e^{2i\delta p} - 1}{2i\delta}, \quad (47)$$

$$A_{jj'} = \frac{L(p_j) - L(p_{j'})}{L(p_j) - L(q_{j'})} \cdot \frac{L(q_j) - L(q_{j'})}{L(q_j) - L(p_{j'})}. \quad (48)$$

It should be noticed that  $L(p)$  is multi-valued. It is easy to check that this soliton solution satisfies the bilinear identity (44).

We impose the reduction condition,

$$\begin{aligned} p_j - q_j &= i^{s-1} (L(p_j)^s - L(q_j)^s) \\ &= (-1)^{s-1} \frac{e^{2i\delta p_j} - e^{2i\delta q_j}}{2i\delta} \\ &= k_j \quad (j = 1, 2, \dots). \end{aligned} \quad (49)$$

Substituting (49) into (45), we have

$$\tau = \sum_{J \subset I} \left( \prod_{j \in J} C_j \right) \left( \prod_{\substack{j, j' \in J \\ j < j'}} A_{jj'} \right) \exp \left( \sum_{j \in J} \left( k_j z + \sum_{n=1}^{\infty} M_n(k_j) t_n \right) \right) \quad (50)$$

for appropriately-defined functions  $M_n(k)$ . We can regard  $z$  and  $t_s$  as the same variable under this reduction. Furthermore, the following lemma holds.

**LEMMA 4.1** *If  $k_j$ 's are real,  $M_n(k_j)$  and  $A_{jj'}$  are also real ( $i = 1, 2, \dots$ ).*

*Proof.* From (48) and (50), we see that

$$L(p_j)^s - L(q_j)^s = (i)^{1-s} k_j, \quad (51)$$

$$L(p_j)^s + L(q_j)^s = (i)^s \left( (-1)^s k_j \cot k_j \delta + \frac{2}{\delta} \right). \quad (52)$$

If  $s$  is odd,  $L(p_j)^s - L(q_j)^s$  is real and  $L(p_j)^s + L(q_j)^s$  is purely imaginary. Hence there exists real  $r, 0 \leq \theta \leq 2\pi$  by which

$$L(p_j)^s = r e^{i\theta}, \quad L(q_j)^s = r e^{i(\pi-\theta)}. \quad (53)$$

If  $s$  is even,  $L(p_j)^s - L(q_j)^s$  is purely imaginary and  $L(p_j)^s + L(q_j)^s$  is real. Then we have

$$L(p_j)^s = r e^{i\theta}, \quad L(q_j)^s = r e^{-i\theta}. \quad (54)$$

For both cases, we choose  $L(p_j), L(q_j)$  as

$$L(p_j) = \sqrt[s]{r} e^{i\frac{\theta}{s}}, \quad L(q_j) = \sqrt[s]{r} e^{i(\pi-\frac{\theta}{s})}. \quad (55)$$

Because  $L(p_j) - L(q_j)$  is defined to be real and  $L(p_j) + L(q_j)$  to be purely imaginary, we see that  $M_j, A_{jj'}$  are real by means of lemma 3.2. This, at the same time, takes care of the multi-valuedness of the function  $L(p)$ .  $\square$

If we consider lemma 4.1 and suppose that  $U(t_s)$  is analytic in the horizontal strip between  $\text{Im } t_s = -i\delta$  and  $\text{Im } t_s = i\delta$ , we can introduce a dependent variable  $u$  which satisfies

$$u = \frac{i}{2}(U(t_s + i\delta) - U(t_s - i\delta)), \quad (56)$$

$$T_s(u) = \frac{1}{2}(U(t_s + i\delta) + U(t_s - i\delta)), \quad (57)$$

where

$$T_s(u(t_s)) = P \int_{-\infty}^{\infty} \frac{1}{2\delta} \cot \left[ \frac{\pi}{2\delta} (\xi - t_s) \right] u(\xi) d\xi. \quad (58)$$

As another concrete example, different from the ILW hierarchy, we apply the preceding argument to the case  $s = 2$ . From (44), we get

$$\oint e^{-i\xi(k, t-t')} (1 - 2\delta k^2) \tau(z + 2i\delta, t - i\epsilon(k^{-1})) \tau(z, t + i\epsilon(k^{-1})) \frac{dk}{2\pi i} = 0, \quad (59)$$

which generates the hierarchy for  $\tau$ . The first few equations are

$$\left( -2D_{t_3} + D_x^3 + 3iD_x D_{t_2} + \frac{3}{\delta} D_x \right) \bar{\tau} \cdot \tau = 0, \quad (60)$$

$$\left( 6D_{t_4} - 4iD_{t_3} D_x - 3iD_{t_2}^2 - iD_x^4 - \frac{12i}{\delta} D_x^2 - \frac{6}{\delta} D_{t_2} \right) \bar{\tau} \cdot \tau = 0. \quad (61)$$

This hierarchy is essentially the same as the 2nd modified KP hierarchy [8]

$$(2D_{t_3} + D_x^3 + 3D_x D_{t_2}) \tau_n \cdot \tau_{n+2} = 0, \quad (62)$$

$$(D_1^4 - 4D_1 D_3 - 3D_2^2 - 6D_4) \tau_n \cdot \tau_{n+2} = 0, \quad (63)$$

$\dots$

The 1-soliton solution for this hierarchy is written in the form

$$\tau = 1 + C \exp[H(t, p) - H(t, q)], \quad (64)$$

$$H(t, p) = (p - q)z + \sum_{n=1}^{\infty} i^{n-1} L(p)^n t_n, \quad (65)$$

$$L(p)^2 = \frac{-e^{2i\delta p} + 1}{2\delta}. \quad (66)$$

By the reduction condition

$$\begin{aligned}
p - q &= i (L(p)^2 - L(q)^2) \\
&= -\frac{e^{2i\delta p} - e^{2i\delta q}}{2i\delta} \\
&= k,
\end{aligned} \tag{67}$$

(64) is reduced to

$$\begin{aligned}
\tau = 1 + \exp &\left[ kz + \sqrt{k \cot k\delta - \delta^{-1} + \sqrt{(k \cot k\delta - \delta^{-1})^2 + k^2}} t_1 + kt_2 \right. \\
&+ \sqrt{k \cot k\delta - \delta^{-1} + \sqrt{(k \cot k\delta - \delta^{-1})^2 + k^2}} \\
&\quad \left. \times \left( -k \cot k\delta + \delta^{-1} + \sqrt{(k \cot k\delta - \delta^{-1})^2 + k^2}/\sqrt{2} \right) t_3 + \dots \right].
\end{aligned} \tag{68}$$

Thus the  $t_2$ -shifts take the place of the  $z$ -shifts. If we impose the analytic condition as before on  $U$ , there exists  $u$  which satisfies

$$u = \frac{i}{2}(U(t_2 + i\delta) - U(t_2 - i\delta)), \tag{69}$$

$$T_2(u) = \frac{1}{2}(U(t_2 + i\delta) + U(t_2 - i\delta)). \tag{70}$$

Now, (60) and (61) are expressible by means of  $u$  as

$$\begin{aligned}
2u_{t_3} - u_{xxx} + 3T_2(u_{xt_2}) - 3(uT_2(u_x))_x + 3u^2u_x \\
- 3\left(u \int_{-\infty}^x u_{t_2} dx\right)_x - \frac{3}{\delta}u_x = 0,
\end{aligned} \tag{71}$$

$$\begin{aligned}
6u_{t_4} + T_2(u_{xxxx}) - 4(uu_x)_x + 12T_2(u)T_2(u)_x + 4u^3u_x \\
- 6(6u^2T_2(u_x))_x + 4T_2(u_{xt_3}) - 4u_x \int_{-\infty}^x u_{t_3} d\xi - 4uu_{t_3} + 3T_2(u_{t_2t_2}) \\
- 6u_{t_2} \int_{-\infty}^x u_{t_2} d\xi + \frac{12}{\delta}T_2(u_{xx}) - \frac{24}{\delta}uu_x - \frac{6}{\delta}u_{t_2} = 0.
\end{aligned} \tag{72}$$

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